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Conformally symmetric nonlinear Schrödinger equation in (1+1) dimensions

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Abstract. We study a (1+1)-dimensional nonlinear Schrödinger equation, which is invariant under dilation and special conformal transformation, in addition to spacetime translations and Galileo boost. The behaviour of a soliton solution to this equation, in an external field which represents a linear repulsive or a harmonic restoring force, is investigated. Explicit solutions are presented. In the former case, the soliton will go to infinity, with its size increasing exponentially with time and, thus, finally collapse. In the latter case, the solution is periodic in time, showing that the soliton binds to the external force. The binding energy is quantized semiclassically by the Bohr–Sommerfeld procedure. In both cases, the centre of the soliton moves in the same way as a classical particle. The nonlinear term in this Schrödinger equation leads to a three-body contact interaction after second quantization. The quantum-mechanical three-body wave equation in configuration space is completely solved.

1. Introduction

Recently, a gauged nonlinear Schrödinger theory was put forward by Jackiw and Pi which describes non-relativistic matter interacting with a Chern–Simons gauge field in (2+1) dimensions [1]. This theory admits conformal symmetries (including dilation invariance and special conformal invariance) in addition to the conventional spacetime symmetries, which include spacetime translations, rotation and Galileo boost. Non-topological soliton solutions carrying electric charge as well as magnetic flux, called Jackiw–Pi solitons, were explicitly obtained in this theory. The behaviour of these solitons in external electromagnetic fields was studied by several authors [2–4]. For uniform magnetic field and harmonic restoring (or linear repulsive) electric field, separately imposed [2, 3] or combined [4], explicit time-dependent solutions are available via time-dependent coordinate transformations. Such time-dependent coordinate transformations were originally used in the study of some linear Schrödinger equations, especially for time-dependent harmonic oscillators [5]. The applicability of these coordinate transformations to the Jackiw–Pi solitons interacting with external fields essentially depends on the conformal symmetries of the Jackiw–Pi theory.

Without coupling to the Chern–Simons gauge field, the field equations in the Jackiw–Pi theory just reduce to the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\nabla^2\psi - g(\psi^*\psi)\psi \tag{1.1}$$

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for the matter field ψ. This is of course much simpler than the original coupled equations. All the symmetries mentioned above are still admitted. If some solution of this equation can be found, its behaviour in external fields can also be realized by the same method used for the Jackiw–Pi solitons. The knowledge of how solitons behave in external fields is important if they are to be applied to condensed-matter physics. However, equation (1.1) is of interest in its own right. Unfortunately, physically useful solutions of this equation have not been discovered at present, as far as we know, though the equation is simpler than that interacting with the Chern–Simons field. In view of this fact, a (1+1)-dimensional counterpart of this equation becomes interesting. In (1+1) dimensions, equation (1.1) takes the form

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\partial_x^2\psi - g(\psi^*\psi)\psi. \tag{1.2}$$

This is the well known nonlinear Schrödinger equation which was solved many years ago [6]. However, this equation does not admit conformal symmetries except for the trivial case $g = 0$. If these symmetries are emphasized, then equation (1.2) is not the (1+1)-dimensional counterpart of equation (1.1) and, hence, does not interest us. In order to reinstate these symmetries in (1+1) dimensions, the nonlinear term in equation (1.2), which is cubic in ψ, has to be modified. It is the purpose of this paper to study such a modified equation in (1+1) dimensions, the nonlinear term of which is quintic in ψ. It reads

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\partial_x^2\psi - 3g(\psi^*\psi)^2\psi. \tag{1.3}$$

This equation admits all the symmetries mentioned before, except rotation invariance which is meaningless in (1+1) dimensions. A somewhat more general nonlinear Schrödinger equation of the form

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\nabla^2\psi - g_0\psi - g_1(\psi^*\psi)\psi - g_2(\psi^*\psi)^2\psi \tag{1.4}$$

was previously studied [7], but in (3+1) dimensions. By the substitution $\psi \to \mathrm{e}^{\mathrm{i}g_0 t}\psi$, (1.4) becomes

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\nabla^2\psi - g_1(\psi^*\psi)\psi - g_2(\psi^*\psi)^2\psi. \tag{1.5}$$

Thus, without loss of generality, one can set $g_0 = 0$. In (1+1) dimensions, (1.5) reduces to (1.2) for $g_2 = 0$, and to (1.3) for $g_1 = 0$. Lie symmetries of equation (1.4) were studied in detail in [7], and classical solutions were investigated there using group-theoretical methods. It turns out that the equation admits Galileo symmetries, including spacetime translations, rotations and Galileo boosts. For either $g_1 = 0$ or $g_2 = 0$ it also contains a dilation invariance. However, this dilation can keep $\int \mathrm{d}\boldsymbol{r}\,\psi^*\psi$ invariant for neither $g_1 = 0$ nor for $g_2 = 0$ in (3+1) dimensions. For this reason, a special conformal symmetry does not exist in (3+1) dimensions in either case. A special conformal symmetry exists for $g_2 = 0$ only in (2+1) dimensions and for $g_1 = 0$ only in (1+1) dimensions. The latter case is to be studied in this paper.

In the next section, we discuss the symmetries of equation (1.3). Though it is a special case of (1.5), we include this section for two reasons. First, in (1+1) dimensions, equation (1.3) admits a special conformal invariance, while in (3+1) dimensions it does not.

Second, [7] gives the symmetry generators in differential operators, while we give them in conserved integrals. We also give the continuity equations with regard to the densities of these conserved generators. This will also make this paper self-contained. In section 3, a soliton solution is discussed. It is time dependent, with the time dependence being in phase. This is somewhat similar to the previously studied soliton solutions in a relativistic charged scalar theory [8]. Though this solution can also be found in [7], we discuss it in some detail since it will be used in the subsequent section. The behaviour of this soliton in an external field is investigated in section 4. The external force may be a linear repulsive or a harmonic restoring force. Exact solutions are acquired by time-dependent coordinate transformations. In the former case, the soliton will generally go to infinity, with its size increasing exponentially with time and, thus, finally collapse. However, there also exists a special case where the soliton will go to the origin (the maximum position of the external potential) and, finally, become a point-like particle. In the latter case, one gets a periodic solution, which shows that the soliton binds to the external harmonic restoring force. The centre of the soliton oscillates around the origin; its size also makes oscillations. The former oscillation has a period twice that of the latter. For both kinds of force, the centre of the soliton moves in the same way as a classical particle. The binding energy of the soliton with the external harmonic force can be quantized semiclassically by the Bohr–Sommerfeld procedure [9]. This is done in section 5. In section 6, we discuss a more general nonlinear Schrödinger equation, where the nonlinear term is of the $(2n+1)$th order in ψ. It reduces to equation (1.2) for $n = 1$ and to equation (1.3) for $n = 2$. A simple soliton solution, similar to the solution discussed in section 3, is easily available. The $n = 2$ case seems quite special in several aspects. Section 7 is devoted to our study of the second quantization of the conformally symmetric nonlinear Schrödinger equation. It turns out that the nonlinear term in the equation gives rise to a three-body contact interaction after second quantization. The quantum-mechanical three-body wave equation in configuration space is completely solved in section 8. This wave equation can be separated into two equations, one of which governs the motion of the centre of mass of the three particles, the other describes their relative motion. The former equation is free, as expected. The latter equation involves the interaction and is of essential interest to us. The energy spectrum of the relative motion turns out to be positive and continuous and, corresponding to a given energy eigenvalue, there exist infinitely many eigenfunctions (denumerable). These are stationary waves and a scattered s-wave. There also exists a single bound state.

2. Action and symmetries

The nonlinear Schrödinger equation studied in this paper arises from the action

$$I = \int \mathrm{d}^2x\, \mathcal{L} \tag{2.1a}$$

$$\mathcal{L} = \mathrm{i}\psi^*\partial_t\psi - \frac{1}{2m}\partial_x\psi^*\partial_x\psi + g(\psi^*\psi)^3 \tag{2.1b}$$

where $\psi(t,x)$ is a classical c-number field in (1+1) dimensions, m is a mass parameter and $g > 0$ governs the strength of the nonlinearity. Throughout this paper, we use natural units where $\hbar = c = 1$. The nonlinear Schrödinger equation derived from (2.1) reads:

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\partial_x^2\psi - 3g(\psi^*\psi)^2\psi \tag{2.2}$$

from which follows the current conservation equation

$$\partial_t \rho + \partial_x J = 0 \tag{2.3}$$

where

$$\rho = \psi^* \psi \tag{2.4a}$$

$$J = \frac{1}{2m\mathrm{i}}(\psi^* \partial_x \psi - \psi \partial_x \psi^*) \tag{2.4b}$$

are the matter density and current, respectively. The total number (or charge)

$$N = \int \mathrm{d}x\, \rho \tag{2.5}$$

is a conserved quantity as a result of (2.3). It generates the phase redefinition $\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$ where α is a real number which leaves action (2.1) invariant. It should be remarked that the difference between (2.2) and (1.2) lies only in the nonlinear term.

The system described by (2.1) possesses more spacetime symmetries than that described by equation (1.2). First, it admits the conventional invariances under spacetime translations and Galileo boost, which are discussed as follows.

2.1. Time translation

If $\psi(t, x)$ is a solution of (2.2), so is $\psi'(t, x) = \psi(t - a, x)$, where a is a constant. The corresponding conserved generator is the Hamiltonian

$$H = \int \mathrm{d}x\, \mathcal{H} \tag{2.6}$$

where $\mathcal{H}$ is the Hamiltonian density, which is also denoted by T^{00}, the time–time component of a non-relativistic energy–momentum tensor:

$$\mathcal{H} = T^{00} = \frac{1}{2m}\partial_x \psi^* \partial_x \psi - g(\psi^*\psi)^3. \tag{2.7}$$

The time independence of H is assured by the continuity equation satisfied by T^{00} and the energy flux T^{01}:

$$\partial_t T^{00} + \partial_x T^{01} = 0 \tag{2.8}$$

where

$$T^{01} = -\frac{1}{2m}(\partial_t \psi^* \partial_x \psi + \partial_x \psi^* \partial_t \psi). \tag{2.9}$$

Equation (2.8) is a consequence of equation of motion (2.2).

2.2. Space translation

If $\psi(t, x)$ is a solution of (2.2), so is $\psi'(t, x) = \psi(t, x - x_0)$, where x_0 is a constant. The corresponding conserved generator is the total momentum

$$P = \int \mathrm{d}x\, \mathcal{P} \tag{2.10}$$

where $\mathcal{P}$ is the momentum density, also denoted by T^{10}, given by

$$\mathcal{P} = T^{10} = \frac{1}{2\mathrm{i}}(\psi^* \partial_x \psi - \psi \partial_x \psi^*) = mJ. \tag{2.11}$$

Note that $T^{10} \neq T^{01}$: the energy–momentum tensor is not symmetric since the system is not Lorentz invariant. Again, the equation of motion results in a continuity equation which ensures the conservation of P:

$$\partial_t T^{10} + \partial_x T^{11} = 0 \tag{2.12}$$

where

$$T^{11} = 2T^{00} - \frac{1}{4m}\partial_x^2 \rho \tag{2.13}$$

is the momentum flux.

2.3. Galileo boost

If $\psi(t, x)$ is a solution of (2.2), so is

$$\psi'(t, x) = \exp[\mathrm{i}mv(x - \tfrac{1}{2}vt)]\psi(t, x - vt)$$

where v is a constant. The corresponding conserved generator G involves the dipole moment

$$G = tP - m\int \mathrm{d}x\, x\rho \tag{2.14}$$

and is obtained from the density

$$\mathcal{G} = t\mathcal{P} - mx\rho. \tag{2.15}$$

The continuity equation

$$\partial_t \mathcal{G} + \partial_x (tT^{11} - xT^{10}) = 0 \tag{2.16}$$

can be easily verified by using (2.3) and (2.12). The conservation of G is thereby established.

Next, we discuss the conformal symmetries. These are not admitted by equation (1.2).

2.4. Dilation invariance

If $\psi(t,x)$ is a solution of (2.2), so is

$$\psi'(t,x)=\frac{1}{\sqrt{|a|}}\psi\left(\frac{t}{a^2},\frac{x}{a}\right)$$

where a is a non-zero constant. Note that this dilation keeps the charge N invariant. In (3+1) dimensions, there also exists a dilation [7], however this dilation does not leave the charge invariant. The corresponding generator

$$D=tH-\tfrac{1}{2}\int \mathrm{d}x\, x\mathcal{P} \tag{2.17}$$

obtained from the density

$$\mathcal{D}=t\mathcal{H}-\tfrac{1}{2}x\mathcal{P} \tag{2.18}$$

is conserved due to the continuity equation

$$\partial_t\mathcal{D}+\partial_x\left[tT^{01}-\frac{1}{2}xT^{11}-\frac{1}{8m}\partial_x\rho\right]=0 \tag{2.19}$$

which follows from (2.8) and (2.12).

2.5. Special conformal invariance

If $\psi(t,x)$ is a solution of (2.2), so is

$$\psi'(t,x)=\frac{1}{\sqrt{|1+at|}}\exp\left[\mathrm{i}\frac{mx^2}{2}\frac{1}{1+at}\right]\psi\left(\frac{t}{1+at},\frac{x}{1+at}\right)$$

where a is a constant. It should be pointed out that this invariance does not exist in (3+1) dimensions [7]. The corresponding generator K involves the quadrupole moment

$$K=t^2H-t\int \mathrm{d}x\, x\mathcal{P}+\tfrac{1}{2}m\int \mathrm{d}x\, x^2\rho \tag{2.20}$$

and is obtained from the density

$$\mathcal{K}=t^2\mathcal{H}-tx\mathcal{P}+\tfrac{1}{2}mx^2\rho. \tag{2.21}$$

Conservation of K is ensured by the continuity equation

$$\partial_t\mathcal{K}+\partial_x\left[t^2T^{01}+\frac{1}{2}x^2T^{10}-txT^{11}-\frac{t}{4m}\partial_x\rho\right]=0 \tag{2.22}$$

which in turn can be verified by employing (2.3), (2.8) and (2.12).

Note that K can also be written as

$$K=-t^2H+2tD+\tfrac{1}{2}m\int \mathrm{d}x\, x^2\rho \tag{2.23}$$

by using (2.17). Then we realize from (2.14), (2.17) and (2.23) that static solutions of (2.2) (if any) for which ρ and $\mathcal{P}$ are time independent should have $H=0$, $P=0$ and $D=0$. This also holds for the solution presented in the next section, since its ρ and $\mathcal{P}$ are time independent, although the solution itself is not.

Finally, we have discrete symmetries. System (2.1) is invariant under the conventional discrete transformations $\psi(t,x)\to\psi(t,-x)$ and $\psi(t,x)\to\psi^*(-t,x)$ induced by space inversion and time reversal, respectively. In addition, we have the following.

2.6. Discrete symmetry

If $\psi(t, x)$ is a solution of (2.2), so is

$$\psi'(t, x) = \sqrt{\frac{a}{t}} \exp\left[\mathrm{i}\frac{mx^2}{2t}\right] \psi\left(-\frac{a^2}{t}, \frac{ax}{t}\right) \tag{2.24}$$

where $a > 0$ is a constant. Since a is not dimensionless we do not scale it to unity by a dilation. However, because of the dilation invariance, a can be regarded as fixed. Thus (2.24) represents a discrete symmetry. Note that this is disconnected with the identity transformation. A similar symmetry is also admitted by the Jackiw–Pi theory [4].

3. Soliton solution

A very simple solution to the nonlinear Schrödinger equation (2.2) can be easily obtained. Although this solution can also be found in [7], we discuss it in some detail since it is to be used in the next section. The simplest solution of (2.2) may have the form

$$\psi(t, x) = \mathrm{e}^{\mathrm{i}\epsilon t} f(x) \tag{3.1}$$

where $f(x)$ is a real function and ϵ is a real constant. If $\epsilon = 0$, it becomes a real static solution. In this case, however, the solution with finite energy is trivial, as can be seen below. Substituting (3.1) into (2.2) yields

$$f'' - 2m\epsilon f + 6mgf^5 = 0 \tag{3.2}$$

where the prime indicates differentiation with respect to argument. From (2.6) and (2.7), we see that the solution with finite energy should satisfy

$$f(x) \to 0 \qquad f'(x) \to 0 \qquad (x \to \infty). \tag{3.3}$$

Integrating (3.2) under condition (3.3) yields

$$f'^2 = 2mf^2(\epsilon - gf^4). \tag{3.4}$$

Since $g > 0$, (3.4) admits non-trivial solutions only when $\epsilon > 0$. If $\epsilon \leqslant 0$, the only solution is the trivial one $f(x) = 0$. When $\epsilon > 0$, integrating (3.4) once again results in the solution

$$f(x) = f_0 \operatorname{sech}^{1/2}\left[\frac{x - x_0}{d}\right] \tag{3.5a}$$

where

$$f_0 = \left(\frac{\epsilon}{g}\right)^{1/4} \qquad d = \frac{1}{2\sqrt{2m\epsilon}} \tag{3.5b}$$

and x_0 is an integration constant. This is obviously a soliton solution since the energy density, as well as the matter density, is localized at any time (actually time independent).

d may be regarded as the size of the soliton, while x_0 stands for the position of its centre. It can be shown for this solution that

$$H = 0 \qquad P = 0 \qquad D = 0 \tag{3.6}$$

as pointed out in the last section. The other conserved quantities can be calculated without much difficulty:

$$N = \frac{\pi}{2\sqrt{2mg}} \tag{3.7a}$$

$$G = -mNx_0 \tag{3.7b}$$

$$K = \tfrac{1}{2}mN\left(x_0^2 + \frac{\pi^2}{4}d^2\right). \tag{3.7c}$$

The above results show that the number N and the energy H are independent of the parameter d (or ϵ) involved in the solution. Nevertheless, solutions with different sizes can not deform into one another under some small perturbation since this is forbidden by the conservation of K. However, stability against arbitrary deformation remains an open question.

The Galileo-boosted solution obtained from (3.1) and (3.5) is given by

$$\psi'(t,x) = f_0 \exp\left[\mathrm{i}\epsilon t + \mathrm{i}mv\left(x - \frac{1}{2}vt\right)\right]\operatorname{sech}^{1/2}\left[\frac{x - x_0(t)}{d}\right] \tag{3.8a}$$

where

$$x_0(t) = x_0 + vt. \tag{3.8b}$$

Equation (3.8b) means that the centre of the soliton moves with a uniform velocity v as expected. The energy of this solution can be found to be

$$H' = \tfrac{1}{2}mNv^2 \tag{3.9}$$

where N is given by (3.7a). Therefore, the soliton has the mass

$$M = mN. \tag{3.10}$$

Other solutions of equation (2.2) are currently under investigation. Results will be reported elsewhere. Quantization of the above soliton also remains to be studied.

4. Interaction with external fields

In this section, we consider the interaction of the field ψ with an external force, which is described by the following action:

$$I = \int \mathrm{d}^2x\, \mathcal{L} \tag{4.1a}$$

$$\mathcal{L} = \mathrm{i}\psi^*\partial_t\psi - \frac{1}{2m}\partial_x\psi^*\partial_x\psi + g(\psi^*\psi)^3 - \frac{1}{2}m\beta x^2\psi^*\psi. \tag{4.1b}$$

The last term in (4.1b) introduces the external force which is either a harmonic restoring or a linear repulsive force according to whether $\beta > 0$ or $\beta < 0$. In the presence of the external field, the only conserved quantities are the number N, given by (2.5), with ρ defined by (2.4a), and the Hamiltonian

$$H = \int \mathrm{d}x \left[\frac{1}{2m}\partial_x\psi^*\partial_x\psi - g\rho^3 + \frac{1}{2}m\beta x^2\rho\right] \tag{4.2}$$

corresponding to invariances of (4.1) under phase redefinition and time translation. All other symmetries except space inversion and time reversal are broken. The equation of motion obtained from (4.1) is

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\partial_x^2\psi - 3g\rho^2\psi + \frac{1}{2}m\beta x^2\psi. \tag{4.3}$$

The continuity equation regarding ρ is given by (2.3) where J is still given by (2.4b).

Solutions of equation (4.3) can be obtained from solutions of (2.2) by the time-dependent dilation transformation [2–5]

$$t \to t' = \int_{t_0}^{t} \frac{\mathrm{d}\tau}{\lambda^2(\tau)} \qquad x \to x' = \frac{x}{\lambda(t)} \tag{4.4a}$$

$$\psi(t,x) \to \psi'(t',x') = \sqrt{|\lambda(t)|}\exp\left[-\mathrm{i}\frac{mx^2}{2}\frac{\dot\lambda(t)}{\lambda(t)}\right]\psi(t,x) \tag{4.4b}$$

where $\lambda(t)$ is a function to be determined below, the overdot denotes differentiation with respect to t and t_0 is an arbitrary constant. Indeed, under transformation (4.4), action (4.1) will be changed to

$$I = \int \mathrm{d}^2x'\,\mathcal{L}' \tag{4.5a}$$

$$\mathcal{L}' = \mathrm{i}\psi'^*\partial_t'\psi' - \frac{1}{2m}\partial_x'\psi'^*\partial_x'\psi' + g\rho'^3 \tag{4.5b}$$

if $\lambda(t)$ satisfies

$$\ddot\lambda(t) + \beta\lambda(t) = 0. \tag{4.6}$$

Note that (4.5) describes a free system, simply the same as (2.1). Equation of motion (4.3) becomes

$$\mathrm{i}\partial_t'\psi' = -\frac{1}{2m}\partial_x'^2\psi' - 3g\rho'^2\psi' \tag{4.7}$$

under (4.4) with (4.6). This can also be derived from (4.5). Once a solution of (4.7) is obtained, one can get a counterpart to (4.3) by the transformation (4.4). We emphasize that the applicability of (4.4) to system (4.1) depends crucially on the conformal symmetries of the corresponding free system (2.1). For the interaction of equation (1.2) with similar external fields, such coordinate transformations do not work. Since a soliton solution of equation (4.7) has been obtained in the last section, we are now in a position to study its behaviour in the external fields. The two different cases $\beta < 0$ and $\beta > 0$ are dealt with separately as follows.

4.1. $\beta < 0$

In this case, the external force is repulsive. Let

$$\beta = -\gamma^2 \qquad (\gamma > 0). \tag{4.8}$$

The general solution of (4.6) is

$$\lambda(t) = c_1 e^{\gamma t} + c_2 e^{-\gamma t} \tag{4.9}$$

where c_1 and c_2 are arbitrary real constants. For convenience, we first discuss the simple case where $c_1 = 1$ and $c_2 = 0$. By choosing $t_0 = +\infty$, we have

$$t' = -\frac{1}{2\gamma} e^{-2\gamma t} \tag{4.10a}$$

$$x' = e^{-\gamma t} x \tag{4.10b}$$

and the solution obtained from (3.1) and (3.5) (of course with t, x replaced by t', x') by transformation (4.4) turns out to be

$$\psi(t,x) = f_0(t) \exp\left[i\frac{\gamma m x^2}{2} - i\frac{\epsilon}{2\gamma} e^{-2\gamma t}\right] \operatorname{sech}^{1/2}\left[\frac{x - x_0(t)}{d(t)}\right] \tag{4.11}$$

where

$$f_0(t) = f_0 e^{-\gamma t/2} \tag{4.12a}$$

$$x_0(t) = x_0 e^{\gamma t} \qquad d(t) = d e^{\gamma t}. \tag{4.12b}$$

Since $x_0(t)$ stands for the position of the centre of the soliton, one sees that the soliton will go to infinity. The size of the soliton $d(t)$ increases exponentially with time. Thus, the soliton will finally collapse. The conclusion is independent of the choice $c_1 = 1$ and it remains correct when $c_2 \neq 0$ since the first term in (4.9) dominates at large t. When $c_1 = 0$, however, things are somewhat exceptional. By the choice $c_2 = 1$ and $t_0 = -\infty$, we have

$$t' = \frac{1}{2\gamma} e^{2\gamma t} \tag{4.13a}$$

$$x' = e^{\gamma t} x \tag{4.13b}$$

and

$$\psi(t,x) = f_0(t) \exp\left[-i\frac{\gamma m x^2}{2} + i\frac{\epsilon}{2\gamma} e^{2\gamma t}\right] \operatorname{sech}^{1/2}\left[\frac{x - x_0(t)}{d(t)}\right] \tag{4.14}$$

where

$$f_0(t) = f_0 e^{\gamma t/2} \tag{4.15a}$$

$$x_0(t) = x_0 e^{-\gamma t} \qquad d(t) = d e^{-\gamma t}. \tag{4.15b}$$

We see that the soliton will finally become a point-like particle and stay at the origin. However, if the Galileo-boosted solution (3.8) instead of (3.1) and (3.5) is used for $\psi'(t', x')$, the soliton will go to infinity, though it will still finally become a point-like particle.

4.2. $\beta > 0$

In this case, the external field represents a harmonic restoring force. Let

$$\beta = \omega^2 \qquad (\omega > 0). \tag{4.16}$$

The general solution of (4.6) is

$$\lambda(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{4.17}$$

where c_1 and c_2 are arbitrary real constants. For simplicity, we choose $c_1 = 1$, $c_2 = 0$ and take $t_0 = 0$. We have

$$t' = \frac{1}{\omega} \tan \omega t \tag{4.18a}$$

$$x' = \frac{x}{\cos \omega t} \tag{4.18b}$$

and

$$\psi(t, x) = f_0(t) \exp\left[\mathrm{i} \left(\frac{\epsilon}{\omega} - \frac{m\omega x^2}{2} \right) \tan \omega t \right] \operatorname{sech}^{1/2} \left[\frac{x - x_0(t)}{d(t)} \right] \tag{4.19}$$

where

$$f_0(t) = \frac{f_0}{\sqrt{|\cos \omega t|}} \tag{4.20a}$$

$$x_0(t) = x_0 \cos \omega t \qquad d(t) = d|\cos \omega t|. \tag{4.20b}$$

We find that both the centre and the size of the soliton oscillate. The former oscillation has a period

$$T = \frac{2\pi}{\omega} \tag{4.21}$$

while the latter has period $T/2$. The maximum value of ρ also varies periodically with time with the same period as $d(t)$ such that the number N is time independent (see below). Other choices of c_1 and c_2 lead to basically the same pattern of motion. The period of the solution is T. This does not depend on the concrete form of $\psi'(t', x')$ and the specific values of c_1 and c_2. In fact, solutions obtained by the transformation (4.4) are always periodic with period T given by (4.21) when $\beta = \omega^2$. This shows that solitons bind to the external harmonic force. The binding energy can be quantized semiclassically. This will be performed in the next section.

We have pointed out that in the presence of the external field the conserved quantities only consist of the number N and the Hamiltonian H. Their expressions are already given in (2.5) and (4.2), respectively. We are now going to evaluate them for the above solutions, for either $\beta < 0$ or $\beta > 0$. It should be remarked that transformation (4.4) does not change the value of N. Thus, the number is still given by (3.7a). As for the Hamiltonian, the result turns out to be

$$H = \tfrac{1}{2} M(\dot{\lambda}^2 + \beta \lambda^2) \langle x^2 \rangle_0 \tag{4.22}$$

where M is the mass of the soliton given by (3.10) and

$$\langle x^2\rangle_0 = \frac{1}{N}\int \mathrm{d}x\, x^2\rho'(x) \tag{4.23}$$

where the subscript zero indicates that this is evaluated in the free system (4.5) and, thus, $\rho'(x)$ is actually determined by the solution (3.1) and (3.5). It is not difficult to find that

$$\langle x^2\rangle_0 = x_0^2 + \frac{\pi^2}{4}d^2. \tag{4.24}$$

In fact, a similar calculation was performed in (3.7*c*). As a consequence of equation (4.6), the Hamiltonian given by (4.22) with (4.24) is time independent, as expected. The energies of the solutions obtained in this section can easily be evaluated by substituting the corresponding solutions of $\lambda(t)$ into (4.22). For example, solution (4.19) with (4.20) corresponds to $\lambda(t) = \cos\omega t$ and has energy

$$H = \frac{1}{2}M\omega^2\left(x_0^2 + \frac{\pi^2}{4}d^2\right). \tag{4.25}$$

In the limit $d \to 0$ ($\epsilon \to +\infty$), this coincides with the energy of a classical harmonic oscillator with frequency ω, mass M and amplitude $|x_0|$ (cf (4.20*b*)).

To conclude this section, we point out that the centre of the soliton moves in the same way as a classical particle. Indeed, in the potential

$$V(x) = \tfrac{1}{2}m\beta x^2 \tag{4.26}$$

a classical particle with mass m obeys the equation of motion

$$\ddot{x} + \beta x = 0. \tag{4.27}$$

It is easy to verify that $x_0(t)$, given in both (4.12*b*) and (4.15*b*), solve (4.27) for $\beta = -\gamma^2$, while that given in (4.20*b*) solves (4.27) for $\beta = \omega^2$. Other choices of constants c_1 and c_2 do not alter this conclusion.

5. Semiclassical quantization

In the preceding section, we have obtained the classical solutions of equation (4.3). When $\beta > 0$, the solution is periodic with period T. This shows that the soliton binds to the external harmonic force. The binding energy can be quantized semiclassically by the Bohr–Sommerfeld condition [9]

$$\int_0^T \mathrm{d}t \int \mathrm{d}x\, \mathrm{i}\psi^*\partial_t\psi = 2\pi n_\mathrm{p} \tag{5.1}$$

where ψ is a periodic solution with period T, not necessarily the specific solution (4.19) with (4.20), and n_p is the 'principal quantum number.' Using (4.1) and (4.2), equation (5.1) leads to the quantization condition for the binding energy E_b:

$$E_\mathrm{b} = \frac{2\pi n_\mathrm{p}}{T} - \langle L\rangle \tag{5.2}$$

where

$$\langle L\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, L \tag{5.3}$$

and L is the spatial integral of (4.1b). Obviously, $\langle L\rangle$, as well as L, depend on the particular classical solution. In order to evaluate L, we substitute (4.3) into (4.1) and obtain

$$L = -2g\int \mathrm{d}x\, \rho^3. \tag{5.4}$$

This cannot be simplified further. Knowledge of specific solutions is necessary for making an ultimate evaluation. For the solution obtained in the preceding section, we have the result

$$L = -\frac{\pi\epsilon}{2\sqrt{2mg}}\frac{1}{\lambda^2(t)} \tag{5.5}$$

where $\lambda(t) = \cos\omega t$, or is given by (4.17) if c_1 and c_2 are not specified. So, we have

$$\langle L\rangle = -\frac{\pi\epsilon}{2T\sqrt{2mg}}\int_0^T \frac{\mathrm{d}\tau}{\lambda^2(\tau)}. \tag{5.6}$$

This is, however, divergent and must be treated carefully. The simplest way to deal with it is by making use of (4.4a), which yields

$$\langle L\rangle = -\frac{\pi\epsilon}{2T\sqrt{2mg}}[t'(T) - t'(0)]. \tag{5.7}$$

This vanishes since t' is a periodic function of t, which can be seen from (4.18a) and can be verified for the more general case. On account of this result, we arrive at

$$E_{\mathrm{b}} = n_{\mathrm{p}}\omega \tag{5.8}$$

where (4.21) has been used. This, indeed, gives the energy levels of a quantum harmonic oscillator with frequency ω, except that the zero-point energy is not included.

6. A more general equation

In this section, we consider a more general nonlinear Schrödinger equation of the form

$$\mathrm{i}\partial_t\psi = -\frac{1}{2m}\partial_x^2\psi - (n+1)g(\psi^*\psi)^n\psi \tag{6.1}$$

where n is a natural number. This coincides with equation (2.2) for $n = 2$ and is basically the same as equation (1.2) for $n = 1$. The equation can be derived from the following action.

$$I = \int \mathrm{d}^2x\, \mathcal{L} \tag{6.2a}$$

$$\mathcal{L} = \mathrm{i}\psi^*\partial_t\psi - \frac{1}{2m}\partial_x\psi^*\partial_x\psi + g(\psi^*\psi)^{n+1}. \tag{6.2b}$$

The Hamiltonian of this system is given by

$$H = \int \mathrm{d}x \left[\frac{1}{2m} \partial_x \psi^* \partial_x \psi - g(\psi^* \psi)^{n+1} \right]. \tag{6.3}$$

System (6.2) is invariant under spacetime translation, Galileo boost, space inversion, time reversal and phase redefinition for general n. Conformal symmetries, however, are admitted only when $n = 2$. In this respect, the $n = 2$ case seems rather special.

A soliton solution of (6.1), similar to that presented in section 3, can easily be obtained. Again, we begin with the ansatz

$$\psi(t, x) = \mathrm{e}^{\mathrm{i}\epsilon t} f(x) \tag{6.4}$$

where $f(x)$ is a real function. For such a simple form of ψ, equation (6.1) reduces to

$$f'' - 2m\epsilon f + 2(n+1)mg f^{2n+1} = 0. \tag{6.5}$$

Integrating (6.5) with the requirement that $f(x) \to 0$, $f'(x) \to 0$ when $x \to \infty$, such that the energy of the solution is finite, one gets

$$f'^2 = 2mf^2(\epsilon - gf^{2n}). \tag{6.6}$$

Since $g > 0$, only $\epsilon > 0$ is acceptable as before. Integrating (6.6) once again results in the solution

$$f(x) = f_0 \operatorname{sech}^{1/n} \left[\frac{x - x_0}{d} \right] \tag{6.7a}$$

where

$$f_0 = \left(\frac{\epsilon}{g} \right)^{1/2n} \qquad d = \frac{1}{n\sqrt{2m\epsilon}}. \tag{6.7b}$$

This is obviously a soliton solution and reduces to (3.5) for $n = 2$. For this solution, one has

$$N \propto \epsilon^{1/n - 1/2} \propto d^{1-2/n} \tag{6.8}$$

where N is the number defined as before. When $n = 2$, the number N is independent of size d or of ϵ (cf (3.7a)). This is not surprising since, for $n = 2$, there is a dilation symmetry by which the solution can be transformed to the one with $\epsilon = 1$, and N is invariant under the dilation. For $n > 2$, the size d increases with number N, which seems natural. For $n = 1$, however, the size of the soliton decreases when the number increases. This perhaps implies that there exists some strong attraction between the matter charges. In regard to the relation between d and N, the case $n = 2$ also exhibits something special.

The energy of solution (6.7) can be expressed in terms of N. In fact, substituting (6.4) into (6.3) gives

$$H = \int \mathrm{d}x \left[\frac{1}{2m} f'^2 - gf^{2n+2} \right]. \tag{6.9}$$

Using (6.6), this reduces to

$$H = \int \mathrm{d}x\,(\epsilon f^2 - 2gf^{2n+2}). \tag{6.10}$$

On the other hand, integrating (6.9) by parts and using (6.5) leads to

$$H = \int \mathrm{d}x\,(-\epsilon f^2 + ngf^{2n+2}). \tag{6.11}$$

Combining (6.10) and (6.11), one arrives at

$$H = \frac{n-2}{n+2}\epsilon N. \tag{6.12}$$

This shows that $H > 0$ for $n > 2$ and $H < 0$ for $n = 1$, while for $n = 2$, the energy vanishes (cf equation (3.6)). Again we find that $n = 2$ is a special case.

For $n = 1$, it is easy to find that

$$N = \frac{1}{g}\sqrt{\frac{2\epsilon}{m}} = \frac{1}{mgd} \tag{6.13}$$

$$H = -\frac{\epsilon}{3g}\sqrt{\frac{2\epsilon}{m}} = -\frac{1}{6m^2gd^3} = -\frac{mg^2}{6}N^3. \tag{6.14}$$

It is remarkable that the soliton becomes point-like ($d \to 0$) when $N \to \infty$. One may expect that in this case the soliton should be very stable against collapse since it possesses a negative infinite energy.

7. The second quantization

In this section, we turn to the second quantization of system (2.1). We consider a quantum field operator ψ and its Hermitian conjugate $\psi^\dagger$, obeying bosonic commutation relations at equal times (a time argument of the operators is suppressed in this section):

$$[\psi(x), \psi(x')] = 0 \qquad [\psi^\dagger(x), \psi^\dagger(x')] = 0 \tag{7.1a}$$

$$[\psi(x), \psi^\dagger(x')] = \delta(x - x'). \tag{7.1b}$$

Time evolution of these operators is governed by the posited Hamiltonian

$$H = \int \mathrm{d}x\,\mathcal{H} \tag{7.2a}$$

$$\mathcal{H} = \frac{1}{2m}\partial_x\psi^\dagger\partial_x\psi - g{:}\,(\psi^\dagger\psi)^3{:}\,. \tag{7.2b}$$

This is similar to the classical Hamiltonian (2.6) with (2.7). However, ordering of non-commutating operators is important in quantum theory, so we have made the normal-ordering prescription of putting all $\psi^\dagger$ to the left of ψ in (7.2b) and denoted it by colons. The Heisenberg equation of motion for ψ reads

$$\mathrm{i}\partial_t\psi = [\psi, H] = -\frac{1}{2m}\partial_x^2\psi - 3g{:}\,(\psi^\dagger\psi)^2{:}\,\psi. \tag{7.3}$$

This is similar to equation (2.2). It should be remarked that we have assumed the bosonic algebra (7.1) to obtain a non-trivial Heisenberg equation. If a fermionic algebra is used, the problem is obviously free. From (7.3) and its Hermitian conjugate, the current conservation equation follows

$$\partial_t \rho + \partial_x J = 0 \tag{7.4}$$

where

$$\rho = \psi^\dagger \psi \tag{7.5a}$$

$$J = \frac{1}{2m\mathrm{i}}(\psi^\dagger \partial_x \psi - \partial_x \psi^\dagger \psi) \tag{7.5b}$$

are the density and current operators, respectively. Thus, the number operator

$$N = \int \mathrm{d}x \, \rho \tag{7.6}$$

is a conserved quantity. In other words, it commutes with Hamiltonian H. Therefore, N and H may have simultaneous eigenstates. We denote these eigenstates by $|E, N\rangle$, where E and N are eigenvalues of the operators H and N, respectively:

$$H|E, N\rangle = E|E, N\rangle \tag{7.7a}$$

$$N|E, N\rangle = N|E, N\rangle. \tag{7.7b}$$

We also posit the existence of the 'vacuum' zero state $|0\rangle$, annihilated by ψ,

$$\psi(x)|0\rangle = 0 = \langle 0|\psi^\dagger(x) \tag{7.8}$$

and also by N and H

$$N|0\rangle = 0 = H|0\rangle. \tag{7.9}$$

However, operating $\psi^\dagger$ on $|0\rangle$ produces another state and, in particular, we define

$$\langle 0|\psi(x_1)\ldots\psi(x_N)|E, N\rangle \equiv u_E^N(x_1, \ldots, x_N). \tag{7.10}$$

It is clear that exactly N field operators are needed to connect $|E, N\rangle$ to $|0\rangle$. $u_E^N(x_1, \ldots, x_N)$ is the N-body wavefunction in configuration space. As a consequence of its definition (7.10) and the commutativity of the operators ψ, it is a bosonic wavefunction, symmetric under position interchange.

Using equations (7.3) and (7.7)–(7.9), one can work out the Schrödinger equation obeyed by the N-body wavefunction u_E^N. Both the one-body and two-body problems turn out to be free

$$\mathrm{i}\partial_t u_E^1 = E u_E^1 = -\frac{1}{2m}\partial_1^2 u_E^1 \tag{7.11}$$

$$\mathrm{i}\partial_t u_E^2 = E u_E^2 = -\frac{1}{2m}(\partial_1^2 + \partial_2^2) u_E^2 \tag{7.12}$$

where $\partial_1 = \partial_{x_1}$ etc. The three-body wave equation, however, involves a three-body contact interaction

$$\mathrm{i}\partial_t u_E^3 = E u_E^3 = -\frac{1}{2m}(\partial_1^2 + \partial_2^2 + \partial_3^2)u_E^3 - 6g\delta(x_1 - x_2)\delta(x_2 - x_3)u_E^3. \tag{7.13}$$

The general N-body ($N \geqslant 3$) wave equation takes the form

$$\mathrm{i}\partial_t u_E^N = E u_E^N = -\frac{1}{2m}\sum_{i=1}^{N}\partial_i^2 u_E^N - 6g\sum_{1\leqslant i<j<k\leqslant N}\delta(x_i - x_j)\delta(x_j - x_k)u_E^N \tag{7.14}$$

as can be expected on the basis of (7.13).

The physical meaning of the Schrödinger equations (7.14) which involve three-body contact interactions, but without two-body contact interactions, remains to be studied further. Nevertheless, the three-body wave equation (7.13) can be completely solved. This will be performed in the final section. We also note that the nonlinear term in equation (6.1) leads to a $(n+1)$-body contact interaction after second quantization.

8. Three-body wavefunction

The task of this final section is to solve the three-body wave equation (7.13). Since we will deal only with the three-body wavefunction, the superscript in u_E^3 will be suppressed. The time-dependence of the wavefunction lies in a factor $\mathrm{e}^{-\mathrm{i}Et}$ as a consequence of the first equality in (7.13). In the following, we omit the factor $\mathrm{e}^{-\mathrm{i}Et}$ in u_E and rewrite equation (7.13) as follows:

$$H_{\mathrm{qm}} u_E(x_1, x_2, x_3) = E u_E(x_1, x_2, x_3) \tag{8.1a}$$

where

$$H_{\mathrm{qm}} = -\frac{1}{2m}(\partial_1^2 + \partial_2^2 + \partial_3^2) - 6g\delta(x_1 - x_2)\delta(x_2 - x_3) \tag{8.1b}$$

is the quantum-mechanical Hamiltonian operator. To solve equation (8.1), we introduce the new coordinates

$$z = \tfrac{1}{3}(x_1 + x_2 + x_3) \tag{8.2a}$$

$$x = \sqrt{\frac{2}{3}}\left(\frac{x_1 + x_2}{2} - x_3\right) \tag{8.2b}$$

$$y = \frac{1}{\sqrt{2}}(x_1 - x_2). \tag{8.2c}$$

Here z is the centre-of-mass position of the three particles, while x and y give their relative positions up to constant factors. The Jacobian determinant of the coordinate transformation is

$$\frac{\partial(x_1, x_2, x_3)}{\partial(z, x, y)} = -\sqrt{3}$$

rather than -1 because the overall coefficient in (8.2*a*) in $1/3$ instead of $1/\sqrt{3}$. In the new coordinates (z, x, y), $H_{\rm qm}$ takes the form

$$H_{\rm qm} = -\frac{1}{6m}\partial_z^2 - \frac{1}{2m}(\partial_x^2 + \partial_y^2) - 2\sqrt{3}g\delta(x)\delta(y). \tag{8.3}$$

It then becomes clear that the variable z can be separated from x and y. In other words, the wavefunction u_E can be factorized:

$$u_E(x_1, x_2, x_3) = u_{\rm c}(z)u(x, y). \tag{8.4}$$

Substituting this equation into (8.1) we obtain two equations:

$$H_{\rm c}u_{\rm c}(z) = E_{\rm c}u_{\rm c}(z) \tag{8.5a}$$

where

$$H_{\rm c} = -\frac{1}{6m}\partial_z^2 \tag{8.5b}$$

and

$$H_{\rm r}u(x, y) = E_{\rm r}u(x, y) \tag{8.6a}$$

where

$$H_{\rm r} = -\frac{1}{2m}(\partial_x^2 + \partial_y^2) - 2\sqrt{3}g\delta(x)\delta(y) \tag{8.6b}$$

with

$$E = E_{\rm c} + E_{\rm r}. \tag{8.7}$$

Equation (8.5) is the wave equation for the centre of mass of the three particles. This is a free problem whose solutions are one-dimensional plane waves. In the following, we mainly deal with equation (8.6), which describes the relative motion of the three particles. This is equivalent to a two-dimensional stationary Schrödinger equation in a δ well potential. It can be rewritten in the form

$$\frac{1}{2m}\nabla^2 u + E_{\rm r}u = -2\sqrt{3}gu(0, 0)\delta(x)\delta(y) \tag{8.8}$$

where ∇^2 is the two-dimensional Laplacian operator. For well-behaved solutions the left-hand side of (8.8) is regular everywhere, whereas the right-hand side is singular at the origin if $u(0, 0) \neq 0$. Therefore, self-consistency demands that

$$u(0, 0) = 0. \tag{8.9}$$

The physical implication of this condition is that the three particles are prevented from coinciding by the three-body contact interaction. (Note that $x = y = 0$ corresponds to

$x_1 = x_2 = x_3$.) On account of this condition, wave equation (8.6) or (8.8) becomes essentially free:

$$\nabla^2 u + 2mE_r u = 0. \tag{8.10}$$

Regular solutions of this equation solve wave equation (8.6) if they satisfy demand (8.9). In the following, we see that all partial waves satisfy (8.9) except the s-wave. However, a solution with $u(0,0) \neq 0$ might be acceptable since the potential is singular at the origin. This will need regularization [10] and a bound state and a scattered s-wave can be obtained.

In order to solve (8.10), with condition (8.9), we introduce the polar coordinates r, θ on the xy plane such that

$$x = r\cos\theta \qquad y = r\sin\theta. \tag{8.11}$$

In these polar coordinates, equation (8.10) takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + 2mE_r u = 0. \tag{8.12}$$

This is solved by

$$u(r.\theta) = R(r)\mathrm{e}^{\pm \mathrm{i}\nu\theta} \qquad \nu = 0, 1, 2, \ldots \tag{8.13}$$

with $R(r)$ satisfying

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left(2mE_r - \frac{\nu^2}{r^2}\right)R = 0. \tag{8.14}$$

If $E_r \leqslant 0$, regular solutions satisfying (8.9) are not available except for the trivial solution $R = 0$. So, we should have $E_r > 0$. Let

$$E_r = \frac{k^2}{2m} \qquad (k > 0). \tag{8.15}$$

The regular solution of (8.14) is

$$R(r) = J_\nu(kr) \qquad \nu = 0, 1, 2, \ldots \tag{8.16}$$

where $J_\nu(kr)$ are Bessel functions of the first kind with argument kr. All these solutions satisfy (8.9), except $J_0(kr)$. The case $\nu = 0$ needs regularization, and will be treated later. Parameter k can take any positive value. So, the energy eigenvalue E_r is continuous according to (8.15). For a given k or E_r, we have the following solutions to (8.10) satisfying (8.9).

$$u_{k\nu}(r,\theta) = J_\nu(kr)\mathrm{e}^{\pm \mathrm{i}\nu\theta} \qquad \nu = 1, 2, \ldots. \tag{8.17}$$

Normalization factors have not been included in (8.17). It should be pointed out that the solutions (8.17) are not bound states, though $u_{k\nu}(r,\theta) \to 0$ when $r \to \infty$. They cannot be normalized in the conventional way.

Though solutions (8.17) satisfy (8.10) and (8.9), they are not the final solutions of our problem since their property under position interchange remains to be studied. (Note that r

and θ are functions of x and y and, hence, functions of x_1, x_2 and x_3.) We learn from the preceding section that the wavefunction $u_E(x_1, x_2, x_3)$ must be symmetric under interchange of the positions x_1, x_2 and x_3. According to (8.4), $u(x, y)$ or $u(r, \theta)$ should also be symmetric under the above position interchange, since $u_c(z)$ is automatically symmetrized (cf (8.2*a*)). Therefore, our task is to symmetrize $u(x, y)$ or $u(r, \theta)$.

From (8.3), (8.5*b*) and (8.6*b*), we have

$$H_r = H_{qm} - H_c. \tag{8.18}$$

On the other hand, from (8.1*b*), one can easily realize that H_{qm} is symmetric under position interchange. That H_c is also symmetric is obvious. Thus, H_r is symmetric under position interchange. On account of this fact, if

$$u(x_1, x_2, x_3) \equiv u(x(x_1, x_2, x_3), y(x_1, x_2, x_3))$$

is a solution to (8.6), so is $u(x_{1P}, x_{2P}, x_{3P})$, where (x_{1P}, x_{2P}, x_{3P}) is any permutation of (x_1, x_2, x_3). Therefore, the symmetrized solution is simply

$$u^s(x_1, x_2, x_3) = \sum_P u(x_{1P}, x_{2P}, x_{3P}) \tag{8.19}$$

multiplied by some appropriate overall factor to satisfy some normalization condition. The summation in (8.19) is over all possible permutations. Thus, it contains six terms. From (8.2), we see that the interchange $(x_1, x_2, x_3) \to (x_2, x_1, x_3)$ corresponds to $(x, y) \to (x, -y)$ or $(r, \theta) \to (r, -\theta)$. On the basis of (8.17) and this relation, the symmetrized solution reads

$$u^s_{k\nu}(r, \theta) = \tfrac{1}{3}[J_\nu(kr)\cos\nu\theta + (x_2 \leftrightarrow x_3) + (x_1 \leftrightarrow x_3)] \tag{8.20}$$

where the factor $\frac{1}{3}$ is included for the following convenience. In order to work out the second term in (8.20), we denote

$$(x'_1, x'_2, x'_3) \equiv (x_1, x_3, x_2)$$

and define z, x', y' and r', θ' in the same way as (8.2) and (8.11). Then, we find that

$$x' = -\frac{1}{2}x + \frac{\sqrt{3}}{2}y \tag{8.21a}$$

$$y' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y \tag{8.21b}$$

or

$$r' = r \qquad \theta' = \frac{2\pi}{3} - \theta. \tag{8.22}$$

We remark that relative Hamiltonian (8.6*b*) is actually invariant under (8.21). For the third term in (8.20), we denote

$$(x''_1, x''_2, x''_3) = (x_3, x_2, x_1)$$

and follow the same method that led to (8.22). We arrive at

$$r'' = r \qquad \theta'' = \frac{4\pi}{3} - \theta. \tag{8.23}$$

Incorporating all these results, we obtain

$$\begin{aligned} u^{\mathrm{s}}_{k\nu}(r,\theta) &= \frac{1}{3} J_\nu(kr)\left[\cos\nu\theta + \cos\nu\left(\theta - \frac{2\pi}{3}\right) + \cos\nu\left(\theta - \frac{4\pi}{3}\right)\right] \\ &= \frac{1}{3}\left(1 + 2\cos\frac{2\nu\pi}{3}\right) J_\nu(kr)\cos\nu\left(\theta - \frac{2\pi}{3}\right). \end{aligned}$$

This vanishes for both $\nu = 3l - 1$ and $\nu = 3l - 2$ where l is a natural number. For $\nu = 3l$,

$$u^{\mathrm{s}}_{k\nu}(r,\theta) = J_{3l}(kr)\cos 3l\theta \qquad l = 1, 2, \ldots. \tag{8.24}$$

By mutiplying (8.24) by a normalization factor $\sqrt{k/\pi}$ and replacing the subscript $\nu (= 3l)$ by l, we obtain the final solutions

$$u^{\mathrm{s}}_{kl}(r,\theta) = \sqrt{\frac{k}{\pi}} J_{3l}(kr)\cos 3l\theta \qquad l = 1, 2, \ldots. \tag{8.25}$$

These are symmetric under position interchange. Given an energy eigenvalue E_{r} or a k, we have infinitely many eigenfunctions $u^{\mathrm{s}}_{kl}(r,\theta)$ numbered by l. (In the following, we have still one more solution.) The eigenfunctions are orthogonal to one another. The wavefunctions corresponding to different energy eigenvalues (or different k's) are also orthogonal to one another. In fact, solutions (8.25) satisfy the orthonormal relation

$$\int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, u^{\mathrm{s}}_{kl}(r,\theta) u^{\mathrm{s}}_{k'l'}(r,\theta) = \delta(k - k')\delta_{ll'}. \tag{8.26}$$

Finally, we return to the case $\nu = 0$. First, we can find a bound state by regularization [10]. Introduce the Fourier transform of $u(\boldsymbol{r})$

$$\phi(\boldsymbol{p}) = \frac{1}{2\pi}\int \mathrm{d}\boldsymbol{r}\, u(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \tag{8.27}$$

with the inverse transform

$$u(\boldsymbol{r}) = \frac{1}{2\pi}\int \mathrm{d}\boldsymbol{p}\, \phi(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \tag{8.28}$$

where $\boldsymbol{r} = (x, y)$ and $\boldsymbol{p} = (p_x, p_y)$. The Fourier transform of (8.8) reads:

$$\left(\frac{\boldsymbol{p}^2}{2m} + B\right)\phi(\boldsymbol{p}) = \frac{\sqrt{3}}{\pi} g u(0,0) \tag{8.29}$$

where $B = -E_{\mathrm{r}}$ is the bound state energy to be calculated. Integrating (8.29) over $\boldsymbol{p}$ and using (8.28), one obtains

$$\int \frac{\mathrm{d}\boldsymbol{p}}{\boldsymbol{p}^2 + 2mB} = \frac{\pi^2}{\sqrt{3}mg}. \tag{8.30}$$

The integral on the left-hand side is divergent. Introducing a cut-off at $p = \Lambda$, we obtain

$$\ln\left(\frac{\Lambda^2}{2mB} + 1\right) = \frac{\pi}{\sqrt{3}mg}. \tag{8.31}$$

For large Λ, this yields

$$B = \frac{\Lambda^2}{2m}\exp\left(-\frac{\pi}{\sqrt{3}mg}\right). \tag{8.32}$$

Since g is an unrenormalized coupling constant, it can be regarded as depending on Λ in such a way that B remains finite when $\Lambda \to \infty$. In this way, we obtain an arbitrary bound-state energy. Replacing the δ potential by a square well of radius r_0, one can also find the bound-state wavefunction [10]. It is not continuous at the origin when $r_0 \to 0$, as expected.

Now we are in the position to solve the case $\nu = 0$. Again, we transform (8.8) to momentum space and get

$$(p^2 - k^2)\phi(\mathbf{p}) = \frac{2\sqrt{3}mg}{\pi}u(0, 0) \tag{8.33}$$

where k is defined in (8.15). The scattering solution of (8.33) can be chosen as

$$\phi(\mathbf{p}) = 2\pi\delta(p_x - k)\delta(p_y) + \frac{2\sqrt{3}mg}{\pi}\frac{u(0, 0)}{p^2 - k^2 - \mathrm{i}\eta} \tag{8.34}$$

where $\eta = 0^+$. Introducing the cut-off Λ once again and integrating (8.34) over $\mathbf{p}$, for large Λ, we have

$$u(0, 0) = 1 + \frac{\sqrt{3}mg}{\pi}u(0, 0)\ln\frac{\Lambda^2}{-k^2 - \mathrm{i}\eta}. \tag{8.35}$$

Using (8.32) to solve (8.35), we get

$$u(0, 0) = \frac{\pi}{\sqrt{3}mg}\left[\ln\frac{E_r}{B} + \mathrm{i}\pi\right]^{-1}. \tag{8.36}$$

Substituting (8.36) into (8.34), we obtain the solution

$$\phi(\mathbf{p}) = 2\pi\delta(p_x - k)\delta(p_y) + 2\left[\ln\frac{E_r}{B} + \mathrm{i}\pi\right]^{-1}\frac{1}{p^2 - k^2 - \mathrm{i}\eta}. \tag{8.37}$$

The corresponding solution in coordinate space is

$$u(\mathbf{r}) = \mathrm{e}^{\mathrm{i}kx} + 2\left[\ln\frac{E_r}{B} + \mathrm{i}\pi\right]^{-1}\int_0^\infty \frac{J_0(pr)p\,\mathrm{d}p}{p^2 - k^2 - \mathrm{i}\eta}. \tag{8.38}$$

This consists of a plane wave plus a scattered s-wave. Indeed, when $r \to \infty$, we have

$$u(\mathbf{r}) \to \mathrm{e}^{\mathrm{i}kx} + \sqrt{\frac{2\pi}{k}}\mathrm{e}^{\mathrm{i}\pi/4}\left[\ln\frac{E_r}{B} + \mathrm{i}\pi\right]^{-1}\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}}. \tag{8.39}$$

The solution involved in (8.38), and linearly independent of (8.17), is

$$u_{k0}(r) = J_0(kr) + 2\left[\ln\frac{E_r}{B} + \mathrm{i}\pi\right]^{-1}\int_0^\infty \frac{J_0(pr)p\,\mathrm{d}p}{p^2 - k^2 - \mathrm{i}\eta}. \tag{8.40}$$

This is the solution corresponding to $\nu = 0$. It depends only on r and, thus, is automatically symmetrized. Therefore, it is readily a solution to our problem of relative motion, which is linearly independent of (8.25). It should be remarked that $u_{k0}(0) \neq 0$.

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References

[1] Jackiw R and Pi S-Y 1990 *Phys. Rev. Lett.* **64** 2969; 1990 *Phys. Rev.* D **42** 3500
[2] Ezawa Z F, Hotta M and Iwazaki A 1991 *Phys. Rev.* D **44** 452; 1991 *Phys. Rev. Lett.* **67** 411; 1991 **67** 1475
[3] Jackiw R and Pi S-Y 1991 *Phys. Rev. Lett.* **67** 415; 1991 *Phys. Rev.* D **44** 2524
[4] Lin Q-G 1993 *Phys. Rev.* D **48** 1852
[5] Takagi S 1990 *Prog. Theor. Phys.* **84** 1019
[6] Zakharov V E and Shabat A B 1971 *Zh. Eksp. Teor. Fiz.* **61** 118 (Engl. transl. 1972 *Sov. Phys.–JETP* **34** 62)
[7] Gagnon L and Winternitz P 1988 *J. Phys. A: Math. Gen.* **21** 1493; 1988 **22** 469
Gagnon L, Grammaticos B, Ramani A and Winternitz P 1989 *J. Phys. A: Math. Gen.* **22** 499
[8] Friedberg R, Lee T D and Sirlin A 1976 *Phys. Rev.* D **13** 2739
[9] Jackiw R 1977 *Rev. Mod. Phys.* **49** 681
Rajaraman R 1982 *Solitons and Instantons* (Amsterdam: North-Holland)
[10] Jackiw R 1991 *MAB Big Memorial Volume* ed A Ali and P Hoodbhoy (Singapore: World Scientific)
Huang K 1982 *Quarks, Leptons, and Gauge Fields* (Singapore: World Scientific); 1989 *Int. J. Mod. Phys.* A 4 1037
Dong S J and Yang C N 1989 *Rev. Math. Phys.* **1** 139